# Frege's unification

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**Abstract**

The purpose of this paper is to examine Frege's views about the *scientific unification* of logic and arithmetic. In my view, what interpreters have failed to appreciate is that logicism is a project of *unification*, not reduction. The notion of unification, I argue, is especially helpful in clarifying how Frege views the projects of *Grundlagen* and *Grundgesetze*, and the differing role of definitions in these works. This allows us to see that there are two types of definition at play in Frege's logicist works. I further use the notion of unification to offer an interpretation of Frege's notion of fruitful definition, which, I think, helps clarify how the two types of definition relate, and how Frege uses them to ground the unification of logic and arithmetic.

## 1 Introduction

Frege's logicism is often presented as the thesis that the laws of arithmetic are analytic. According to a particularly influential interpretation, this is an *epistemological* thesis about the nature of arithmetical knowledge. The idea being that Frege's project was to *reduce* arithmetic to logic and, in so doing, to show that arithmetical truths are analytic and, hence, knowledge of them a priori. For this reduction to succeed, Frege required definitions of the core arithmetical concepts, beginning foremost with an explicit definition of number. It appears, however, that Frege's definitions are unable to underwrite the claim that the *Grundgesetze* derivations show that *arithmetical* truths are analytic. Arithmetical truths, presumably, are truths *about numbers*. But Frege's defined concepts do not appear to express *arithmetical content*, and consequently, it is unclear how arithmetical content is preserved in the mathematical "reduction" of arithmetic to logic.[1] This lack of clarity underwrites the basis of Benacerraf's argument that logicism was *not* an epistemological thesis for Frege [Benacerraf(1981)].

Benacerraf's argument has inspired much discussion (and dispute!) in the literature.[2] Problematically, its conclusion appears to be diametrically opposed to several of Frege's own explanations of his project. In my view, what interpreters have failed to appreciate is that logicism is equally a thesis about *logic*, in particular, a thesis about the *expressiveness* of logic. Once we take this into consideration, this raises at once the following two questions: (a) *What content do arithmetical truths express?* and (b) *Is this content derivable within logic?* This second question entails the further question of whether the principles of logic can underwrite the existence of mathematical objects. When logicism is cast as a thesis about *logic*, the central task of Frege's formal derivation of arithmetic within logic is to defend a positive answer to (b), rather than an answer to (a). Furthermore, this derivation would not just show that the truths of arithmetic are reducible to logic, but rather it would show that logic and arithmetic "constitute a unified science" [Frege(1885), 112].

[1] *Grundgesetze*, i.e., Gottlob Frege's *Grundgesetze der Aritmetik* (1903). Throughout this paper, references are to Ebert and Rossberg's translation, i.e., [Frege et al.(2013)Frege, Ebert, and Rossberg]. I shall also use "*Gg*" as an abbreviation. Similarly, I shall abbreviate *Die Grundlagen der Arithmetik* as "*Grundlagen*" or "*Gl*". All references are to Austin's translation, i.e., [Frege and Austin(1980)].

[2] See [Blanchette(1994), Weiner(1984), Tappenden(1995), Jeshion(2001)].

The purpose of this paper is to examine Frege's views about the *unification* of logic and arithmetic. The notion of unification, I argue, is especially helpful in clarifying how Frege views the projects of *Grundlagen* and *Grundgesetze*, and the differing role of definitions in these works (sections 2 and 3). I use the notion of unification to offer an interpretation of Frege's notion of fruitful definition, which, I think, helps clarify how the two notions of definition relate (section 4). Finally, I use the foregoing discussion to address our opening question (section 5): Is the *Fregean* thesis that arithmetical truths are analytic an *epistemological* thesis?

## 2 Logic and arithmetic as a unified science

Frege clarifies his view of the relationship between logic and arithmetic in the paper "On Formal Theories of Arithmetic" ([Frege(1885)]; hereafter [FTA]), which followed the publication of *Grundlagen*. In [FTA], he presents the logicist thesis as the thesis that logic and arithmetic are a unified science:

> [N]o sharp boundary can be drawn between logic and arithmetic. Considered from a scientific point of view, both together constitute a unified science. [112]

Frege's view is that, from a scientific perspective, there are no relevant distinctions between logic and arithmetic, viz., the domain and the inference rules of arithmetic are part of logic, and arithmetical concepts are definable in (and hence reducible to) logic. The passage continues:

> If we were to allot the most general basic propositions and perhaps also their immediate consequences to logic while we assigned their further development to arithmetic, then this would be like separating a distinct science of axioms from that of geometry.

In *Grundlagen*, Frege argued that arithmetic has the same domain as logic on the grounds that arithmetical principles, like logical laws, govern everything thinkable. This is based on the conception of logic as *universal*, viz., as governing the domain of conceptual thought [Goldfarb(2001)]. On this conception, logical laws express (substantive) truths about any subject matter and these laws are, therefore, fully general.[3] Moreover, since every object of (conceptual) thought can be counted, there appears to be no special domain of arithmetical objects.[4] Frege uses this conclusion to argue that concepts have numbers (i.e., that a statement of number is a claim about a concept) and, more specifically, that numbers are extensions of concepts.[5] In *Grundlagen*, he also underlines the analogy between the relationship of the truths of arithmetic to the truths of logic and the relationship of the theorems of geometry to its axioms: "The truths of arithmetic would then be related to those of logic in much the same way as the theorems of geometry to the axioms" [*Gl*, §17]. These considerations suggest that arithmetic is part of logic (just like the theorems of geometry are part of geometry). If this is correct, Frege says, then the principles of logic underwrite the truths of arithmetic, which means that we can express *arithmetical content* in purely logical terms. It also means that we can show that arithmetical truths are theorems of logic.

[3] Frege's conception of logic can be contrasted with what Goldfarb calls a *schematic* conception of logic, according to which logical laws are schemata, i.e., formulas that are only partially interpreted.

[4] In [FTA] Frege again observes that "just about everything that can be an object of thought" can be counted, from which he draws the same conclusion.

[5] For example, Frege analyzes a statement such as "There are eleven houses on 7th street" as the claim that the concept ¡¡house on 7th street¿¿ is satisfied by eleven objects. It is thus a statement about the cardinality of a concept.

Frege's view is that the formal unification of arithmetic and logic requires two steps: first, the *reduction* of arithmetical concepts by means of definitions, and in addition, the *derivation* of arithmetic within logic. The first step is to show that arithmetical content can be expressed in purely logical terms, whereas the second step is to show that arithmetical truths are theorems of logic. In this line, the project of *Grundlagen* is the *discovery* of the content of arithmetic. The development of arithmetic in *Grundgesetze* builds on these results, however, its project is the *justification* of that content.

***Grundlagen*: Discovery** The *conceptual basis* for Frege's *Grundlagen* definition of Number is the thesis that arithmetic is a branch of logic. Part of his argument for this thesis is that numbers can be identified as extensions of concepts. Given Frege's explicit definition of Number, numbers are extensions of second-level concepts.[6] The definition is intended to show that numbers can be described in purely logical terms. This shows how arithmetical content can be reduced to logical content, i.e., that arithmetical content can be expressed logically.[7]

Extensions of concepts are *logical* objects, i.e., objects whose existence can be inferred on purely logical grounds.[8] The claim that arithmetical propositions can be seen to express truths about *those* objects is based on Frege's arguments for the conception of numbers according to which (a) concepts have numbers, (b) numbers are (abstract) objects and (c) these objects satisfy Hume's Principle (i.e., the principle that equinumerous concepts have the same number).[9] Apart from (c), these theses are based on the presupposition that arithmetical propositions express truths about the domain of logic, and not about some more restricted domain (e.g., the Kantian domain of the intuitable).[10] To show that logic and arithmetic are a unified science, Frege has to further show that "there is no peculiar arithmetical mode of inference that cannot be reduced to the general inference-modes of logic" [Frege(1885), 113]. For example, the principle of induction might be an extra-logical inference rule. If it turns out that the proofs of the basic propositions of arithmetic require an extra-logical mode of inference, then the whole approach is undermined. Frege's view, of course, is that we "have no choice but to acknowledge the purely logical nature of the arithmetical modes of inference" [113].[11] But he also thinks that this requires proof [*Gl*, §1].

The arguments from *Grundlagen* are, therefore, not sufficient to show that arithmetic can be *unified* with logic. Frege has only offered a logicist account of how we come to discover the logical means by which we can express the content of arithmetical truths. But discovering the content expressed by these truths is not sufficient for their *justification*. Indeed, as is well-known, Frege separates the context of discovery from the context of justification: "It not uncommonly happens that we first discover the content of a proposition, and only later give the rigorous proof of it, on other and more difficult lines" [*Gl*, §3]. Note too that this passage is part of Frege's

[6] Here is the definition: "The Number which belongs to the concept $F$ is the extension of the concept "equal to the concept $F$"" ( *Gl*, §68).

[7] Notice that this does not show that the numbers *are* extensions of concepts because it is only reductive over content.

[8] At least, ignoring for the moment the inconsistency of Basic Law V.

[9] To be more precise: what is now known as Hume's Principle is the condition that the number of $F$s is equal to the number of $G$s if and only if there is a one-to-one correspondence between the $F$s and the $G$s.

[10] Here (c) encodes the idea that numbers are measures of cardinality and are used for counting. Frege analyzes the notion of cardinal number in terms of the equinumerosity of concepts, such that two concepts are equinumerous if their extensions can be placed in a one-to-one correspondence.

[11] As he explains, "[i]f such a reduction were not possible for a given mode of inference, the question would immediately arise, what conceptual basis we have for taking [the mode of inference] to be correct" [113]. The other options he considers are Kantian "intuition" and observation, and, as he has already argued in *Grundlagen*, neither of these options is tenable.

discussion of the analytic/synthetic and a priori/a posteriori distinctions. For Frege, these distinctions concern the justification of a proposition, rather than its content. Analogously, his logicist view is that an account of the content expressed by arithmetical propositions is not sufficient for their justification, for justification requires proof.

In the conclusion of *Grundlagen*, Frege explains that he hopes "to have made it *plausible*" that the laws of arithmetic are analytic [§87; my emphasis]. Given his notion of analyticity, this means that these laws can be proved using only logical laws and definitions, and thus that arithmetic "becomes simply a development of logic, albeit a derivative one" [§87]. To support this thesis, Frege has offered an explicit definition of cardinal number, and sketches for the proofs that the numbers, as characterized by this definition, have the properties of the natural numbers (see §§70-73). To raise this thesis from plausible to justified, however, requires gap-free derivations of the laws of arithmetic from pure logic. For as long as Frege has not shown that these laws (with their meaning settled as in *Grundlagen*) can be derived within a system of pure logic, it can still be denied, as presumably Kantians would have, that logic can ground the truths of arithmetic.

***Grundgesetze*: Justification** Frege thinks that if arithmetical laws are truths about logical objects (per his analysis in *Grundlagen*) then these laws must be provable in pure logic, and so only by providing such proofs can he vindicate his logicist analysis of Number. As he explains in the foreword of *Grundgesetze*:

> By this act I aim to *confirm* the conception of cardinal number which I set forth in the latter book. The basis of my results is articulated there in §46, namely that a statement of number contains a predication about a concept; and *the exposition here rests upon it*. [*Gg* vol. 1, viii-ix; my emphasis]

At issue is not *what* content sentences of arithmetic express, but rather *whether* these sentences, with their content already settled, are derivable within a system of pure logic.[12]

In *Grundgesetze*, Frege shifts to talk of the "ideal of a rigorous scientific method", according to which proof is constructed in an axiomatic system. This shift corresponds to the shift from "discovering" the content of arithmetical claims (in *Grundlagen*) to that of their justification (in *Grundgesetze*). For Frege, the firmest type of justification is logical proof in an axiomatic system. Such a system, on this view, consists of the complete specification of a language, together with axioms (formulated in that language), inference rules and possibly definitions. Questions about justification, then, can only be treated rigorously in the context of a system, i.e., an entire theory. This also means that whether a proposition is analytic depends on the system in which it is proved.[13]

The task of *Grundgesetze*, then, is explicitly to address the shift from claims about discovery to demonstrations of justification. Its introduction opens thus:

> In my *Grundlagen der Arithmetik* I aimed to make it plausible that arithmetic is a branch of logic. . . In the present book this is now to be *established by deduction* of the simplest laws of cardinal number *by logical means alone*. [*Gg* vol. 1, 1; my emphasis]

[12] As Frege explains: "Usually, mathematicians are merely concerned with the content of a proposition and that it be proven. Here the novelty is not the content of the proposition, but how its proof is conducted, on what foundation it rests" [*Gg* vol. 1, viii].

[13] See also [Dummett(1991)] for discussion of this point.

Frege assumes that the *Grundgesetze* theorems that are labeled "basic laws of cardinal number " are just concept-script renderings of the basic propositions of arithmetic. According to the *Grundlagen* account, the natural numbers are *cardinal* numbers and thus the basic propositions of arithmetic are the basic laws of cardinal number.[14]

## 3 Frege's definitions

Prior to *Grundlagen*, Frege briefly discusses definitions in *Begriffschrift*.[15] Definitions, he says, are just stipulations that serve to introduce abbreviations into a language:

> . . . nothing follows from [a definition] that could not be inferred without it. Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation. They serve besides to emphasize a particular combination of signs in the multitude of possible ones, so that our faculty of representation can get a firmer grasp of it. [Frege(1879), 55]

Frege states definitions as identities, viz., sentences of the form $(a = b)$.[16] Once so stated, a definition immediately turns into an analytic judgment, and, " [s]o far as the derivations that follow are concerned, [it] can therefore be treated like an ordinary judgment" [Frege(1879), 55]. The only *content* expressed by this judgment is trivial: it is an instance of the law of identity $(a = a)$. *From the perspective of logical proof*, definitions are, therefore, redundant.[17] Consequently, Frege requires definitions to be eliminable and non-creative.[18] A definition is *eliminable* when its defined term can be eliminated in favor of its defining phrase in any sentence of the language. Eliminable definitions are such that, in Frege's words, "if the *definiens* occurs in a sentence and we replace it by the *definiendum*, this does not affect the thought at all" [Frege(1914), 208]. A definition is *non-creative* when it is *only* used to stipulate the meaning of a term, and cannot, therefore, help prove any result that could not already be proved prior to its introduction.[19]

Where does this leave the role of definitions in Frege's logicist project? Definitions, it seems, must play two distinct roles. First, in *Grundlagen*, where the goal is to show that arithmetical content can be reduced to logic, Frege needs to provide logical definitions of arithmetical concepts. These definitions must *specify a content* in a way that makes plausible the justification of arithmetic. Second, in *Grundgesetze*, where the goal is justification, Frege needs to

[14]See also [Frege et al.(2013)Frege, Ebert, and Rossberg, vol. 2, 155-6].

[15]*Begriffschrift*, i.e., *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought* (1879). References are to van Heijenoort's translation, i.e., [Frege(1879)].

[16]Frege's view of identity shifts from a metalinguistic view in *Begriffschrift* to an objectual view in *Grundgesetze*. It has been argued that in *Begriffschrift*, Frege uses the "=" sign for coreference. Though for a detailed argument against this interpretation, see [May(2012)].

[17]Frege repeats this on several occasions. For example, in "Logic in Mathematics" he writes that it "appears from this that definition is, after all, quite inessential. In fact considered from a logical point of view it stands out as something wholly inessential and dispensable" [Frege(1914), 208]. And in "Foundations of Geometry: First Series" he writes: "[A definition] is only a means for collecting a manifold content into a brief word or sign, thereby making it easier for us to handle. This and this alone is the use of definitions in mathematics" [Frege and MacGuinness(1984), 274].

[18]Note, however, that his discussion of these requirements only occurs in his later work. Also, though Frege thinks that definitions must be eliminable, he does not use this terminology.

[19]This notion corresponds to the familiar notion of conservativeness, according to which a definition is conservative when it cannot help prove any theorem (not involving the defined term) that would otherwise be unprovable. See [Belnap(1993)] for discussion of the eliminability and conservativeness criteria, and [Boddy(manuscript)] for further discussion of these criteria in Frege's work.

show that the *Grundlagen* definitions can be added as conservative extensions to a system of pure logic, and can be used, subsequently, to derive the basic laws of cardinal number. These two roles suggests that Frege has two notions of definition at play: definitions that arise from (conceptual) analysis, and the *Begriffschrift* notion of definition as mere abbreviation.

***Grundlagen*: Conceptual analysis** It appears, then, that the *Grundlagen* definitions, being the result of Frege's logical-philosophical analysis of arithmetical concepts, are *not* conventions of abbreviation and are, therefore, not expected to be eliminable. These definitions, qua logical definitions of arithmetical terms, must preserve (at least part of) the meaning of their defined terms, terms which are not *new* but already have an established use in mathematical practice. Indeed, these definitions appear to be more akin to what Frege in later work calls "analytic definitions" [Frege(1914)]. An "analytic definition" is the result of a logical analysis of a term "with a long established use" that already has a sense, whose sense is analyzed into a complex expression.[20] Such a "definition" is not an arbitrary stipulation and is, therefore, *not* a definition in the *Begriffschrift* sense at all but "is really to be regarded as on axiom" [210]. Frege contrasts analytic definitions with "definition" *tout court*. These are the familiar type of definitions from *Begriffschrift*. This leaves us asking, however, did Frege regard the *Grundlagen* definitions as proper definitions?

The answer is "yes", but with a caveat. The *Grundlagen* definitions are explanations of the meaning of terms that are intended to be used as eliminable definitions in Frege's *Grundgesetze* proofs. Indeed, the *Grundgesetze* definitions are essentially just the *Grundlagen* definitions [Heck(1993), 269]. The caveat is that definitions are properly speaking only definitions in the context of a particular theory (i.e., what Frege calls a "system"). Frege of course intends to add the *Grundlagen* definitions to his logical system. *For this purpose*, it is only relevant whether these definitions are eliminable and non-creative with respect to *that* system.

Similarly, definitions are *stipulations* about the meaning of new terms *within a system*. The terms introduced via definitions need only be *new to the system*. For example, Frege can introduce a number operator into his logicist system with a stipulative definition but, he says, "if we do this, we must treat it as an entirely new sign which had no sense prior to the definition", and that "[i]n constructing the new system we take no account, logically speaking, of anything in mathematics that existed prior to the new system" [Frege(1914), 211]. The choice for a particular new term is "arbitrary" in that it is only constrained by the rules for the construction of well-formed expressions in the language and the requirement that the term be new (to the language).[21] It does not follow that Frege's choice for his defined terms is arbitrary. The *Grundlagen* definitions, being "analytic" (in the relevant sense), are not at all arbitrary. For these definitions must *specify a content* in a way that allows for the *justification of arithmetic*. This does not preclude these definitions from being *used as* eliminable definitions in *Grundgesetze*, however. For the sake of exposition, I shall call the *Grundlagen* definitions "conceptual definitions," and definitions used to introduce abbreviations (as found in *Begriffschrift* and *Grundgesetze*) "proper definitions".[22]

[20] Thus, Frege explains, we start from "a simple sign with a long established use" and then "give a logical analysis of its sense, obtaining a complex expression which in our opinion has the same sense" [210].

[21] See §§26-28 and 33 of *Grundgesetze* (vol. 1) for the formation rules for names (i.e. terms) in the concept-script, and the rules for constructing definitions. Frege here presents the following "governing principle for definitions: Correctly formed names must always refer to something." This is followed by (what is now known as) the proof of referentiality for the *Grungesetze* names.

[22] To be clear: my use of the term "conceptual definition" differs from Frege's use of the term "analytic definition".

***Grundgesetze*: Gap-free proof** Frege thinks that a successful justification of a reductive analysis of number should result in a definition of number that can be added as a conservative extension to its reducing theory. Moreover, he insists that to *prove the worth of the definition*, it must be shown that it enables the construction of gap-free proofs of the well-known properties of the numbers, as described by the laws of cardinal number. The central task of *Grundgesetze* is the gap-free proof of these laws in Frege's logical system. Hence, Frege returns to the *Begriffschrift* conception of definition. The only difference being that in the *Grundgesetze* definitions, the defined phrase is stipulated to have the same sense and the same reference as the defining phrase.[23] It is not just the justification of the laws of cardinal number that is at issue, Frege also intends to justify his *Grundlagen* definitions. For Frege, the justification of a definition "must be a matter of logic" [*Gl*, ix]. Specifically, the logical justification of a definition consists in the definition satisfying the eliminability and non-creativity requirements.

As used in *Grundgesetze*, the explicit definition of number cannot show that any of the derived theorems are indeed theorems *of arithmetic*. But what it can help show is that logic is sufficiently expressive for the proofs of the laws of cardinal number, such that these laws are already recognized as *arithmetical*.[24] For these proofs show that the derived sentences are grounded on principles of logic only. The definition, being constructed in a language whose primitive vocabulary is purely logical, does not express any non-logical content. It thus shows that no *additional*, non-logical, content is required for the proofs of the laws of cardinal number. In addition, it helps facilitate the recognition of the numbers in the logicist development of arithmetic.

# 4 *Grundlagen*'s fruitfulness requirement of definitions

As we have seen, there must be an appropriate tie between the two types of definition such that the *Grundgesetze* development of arithmetic can justify the *Grundlagen* conception of cardinal number. It would be a mistake, then, to view the *Grundgesetze* definitions as conceptual definitions, as [Horty(2007)] proposes.[25] Frege thinks that once we present an axiomatic system that is constructed "from the bottom up", like *Grundgesetze*'s system, there is *no need* for conceptual definitions because we can treat the defined terms as entirely new. The task of *Grundgesetze* is the *justification* of (arithmetical) content, not its discovery. Prior to the explicit definition of number, Frege has already concluded, on the basis of his conceptual definition of Number in *Grundlagen*, that numbers can only be logical objects. In *Grundgesetze*, he takes this conception of the numbers for granted.[26]

How, then, can Frege's definitions play their two roles? Frege's answer, in my view, is that to play both roles, definitions must be fruitful. According to Frege, the *Grundlagen* definitions have (logicist) worth only in so far as they allow for the gap-free proof of the laws of cardinal number. If his definition of Number cannot be used to this end, he says, then it "should be rejected as completely worthless" [*Gl*, §70]. In *Grundlagen*, this worth is witnessed by the requirement that definitions be *fruitful*, such that definitions are fruitful when their introduction is necessary for the gap-free proof of the *sentences* in which their defined terms occur.[27] Notice

[23] *Begriffschrift*, of course, predates Frege's bifurcation of meaning into sense and reference.

[24] That is, per *Grundlagen*, these laws express the scientific content of the basic propositions of arithmetic.

[25] According to Horty, Frege's definitions play their two roles *simultaneously*, viz., to introduce abbreviations into the language of logic and to explicate expressions already in use in the language of arithmetic.

[26] See e.g., [*Gg.* vol. 2, 153]. It is of course undermined by Russell's paradox.

[27] In Frege's most explicit formulation of the requirement, he clarifies the notion of fruitful definition as follows: "Those [definitions] that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless" [*Gl*, §70].Frege's notion of fruitful definition has engendered

that, in the case of the definition of number, these sentences are, foremost, the laws of cardinal number.

Frege's view is that definitions should be fruitful because by being fruitful, definitions show that no *additional* content is required for the proofs of the sentences in which their defined terms figure. This shows that these sentences express the content that they are afforded by the definition. That is, fruitful definitions underwrite the proofs of the sentences in which their defined terms occur, and these proofs show that the defined terms are used in these sentences *exactly with their stipulated meaning*. Frege's *Grundgesetze* proofs of the laws of cardinal number confirm that his definition of Number specifies a content in a way that allows for the derivation of arithmetic.[28] It also confirms that the definition identifies *logical* objects as the numbers. Now, only some of the *Grundgesetze* definitions are paired with a conceptual definition (from *Grundlagen*). Whenever there is such a pairing, the fruitfulness of the definition justifies its analytic counterpart, and demonstrates the sense in which the *Grundgesetze* offers a logical development of *arithmetic*.

While Frege initially discusses the fruitfulness requirement in *Grundlagen*, he continues this in "Logic in Mathematics", where he compares unfruitful definitions to stucco-embellishments on buildings and says that, like such embellishments, unfruitful definitions are only "ornamental" and play no role in the actual development of arithmetic. Such presumed definitions are not really definitions, he says, as they do not actually fix the reference of the numerals but are "only included because it is in fact usual to do so" [212]. As in *Grundlagen*, at issue is the *worth* of the definition in underwriting (or justifying) an analysis of its definiendum [Frege(1914)].[29] According to Frege, if the definition of Number is not shown to underwrite the proofs of arithmetical theorems, then it is also not shown that these theorems express truths about the objects identified by the definition as the numbers. In this case, the definition is useless as a conceptual definition and useless as a proper definition.

## 5 Are ordinary arithmetical truths analytic?

As noted, Benacerraf has argued that the *Fregean* thesis that arithmetical truths are analytic is not an *epistemological* thesis. The basic idea, as expressed in [Benacerraf(1981)], is that if logicism is an *epistemological* project, then Frege's definitions must preserve the "ordinary" meaning of their defined terms because only when the definitions express arithmetical content can the logicist derivation of *Frege's* arithmetic underwrite the thesis that the truths of *ordinary* arithmetic are analytic, and hence yield a priori knowledge.

Benacerraf's contention is that "Frege did *not* expect *even reference* to be preserved by his definitions" [29].[30] Benacerraf observes that Frege allows that there are several ways of reducing arithmetic to logic and, in particular, that he suggests that the numbers need not be

much discussion in the literature. See, e.g., [Benacerraf(1981), Boddy(manuscript), Horty(2007), Shieh(2008), Tappenden(1995), Weiner(1990)].

[28]Note that this content is specified by the definiens of the definition, and so the fruitfulness of Frege's definition of Number depends on whether it can be shown that the referents of the definiens have the well-known properties of numbers.

[29]In [Frege(1914)] Frege does not use the word "fruitful" (*fruchtbar*) though he discusses the same requirement. As [Tappenden(1995)] observes, Frege no longer uses the "fruitful definitions" terminology in his post-1884 writings. The fruitfulness condition is also not among (or implied by) the principles of definition listed in §33 of *Grundgesetze*. However, each of the *Grundgesetze* definitions is fruitful (in the above sense).

[30]This claim should not be confused with Benacerraf's claim, from [Benacerraf(1965)], that in reductionist projects, like Frege's, the definitions of arithmetical terms do not preserve their ordinary meaning (or that meaning does not determine reference) because there are different ways of assigning referents to the mathematical vocabulary.

identified with extensions of concepts, but could have been identified with different referents. This shows, he argues, that Frege's definitions are not expected to preserve the referents of the numerals of ordinary arithmetic.[31] If so, then the sentences derived from those definitions are not expected to express truths of *ordinary* arithmetic either. Hence he concludes that Frege did not intend to show that arithmetic is analytic, and thus yields a priori knowledge.

Benacarraf's argument has inspired much discussion in the literature. Interpreters have focused on the question of whether Frege's project can nonetheless still be taken as an *epistemological* project. Against Benacerraf's interpretation, Frege himself repeatedly says that his concern is with the nature of our knowledge of arithmetic. In this line, [Weiner(1990)] argues that Frege intended to present a theory that was to *replace* ordinary arithmetic because, on his view, the numerals of ordinary arithmetic did not refer prior to his work.[32] There was, therefore, no reference to be preserved by Frege's definitions. Her response to Benacerraf is that Frege's logicist arithmetic can replace ordinary arithmetic because "it has all the applications of arithmetic" [115].

If "ordinary" arithmetic is number theory as "ordinarily understood", where this requires some shared view of the content expressed by the numerals, then Frege does not think that there is an ordinary arithmetic. Hence, there is also not an ordinary arithmetic *to replace*. Indeed, his opening argument in *Grundlagen* is exactly that there is no common understanding of what numbers are. The problem that underlies the *Grundlagen* discussion is exactly that it is not clear *exactly what content* the definitions of arithmetical concepts must preserve. That is, the problem is not that the "ordinary" numerals do not have referents prior to Frege's work, but rather that it is *unclear* what exactly these referents are: most mathematicians have some notion of what numbers are—enough to agree on the truth of arithmetical claims—but this "inkling" is imprecise and unarticulated and, therefore, defective [Frege(1914), 221]. Frege's view is that to correct this shortcoming, he needs to derive the well-known properties of the numbers from his definition of Number. The development of arithmetic in *Grundgesetze* justifies the thesis that principles of pure logic can found arithmetical content. Moreover, what *Grundgesetze* shows is that the justification of Frege's definitions, qua definitions of arithmetical concepts, ultimately depends on their ability to help prove the laws of cardinal number within such a system.[33] If this is correct, then *Frege's* arithmetic is intended to be just *a scientifically founded version of ordinary arithmetic*. So clearly logicism is an epistemological thesis. This was, of course, Dummett's point, and I agree [Dummett(1991)]. However, what interpreters, including Dummett and Benacerraf, have failed to appreciate is that the *Grundgesetze* is primarily a work of *justification*, and together with *Grundlagen*, that logicism is a project of unification, not reduction.

In *Grundlagen*, Frege says that the investigation into the foundations of arithmetic is "a task which is common to mathematics and philosophy" [*Gl*, xviii]. I hope to have shown that he means this quite literally. For Frege, to found arithmetic is nothing less than to undertake the *unification* of these two sciences. This requires two steps: the philosophical (or conceptual) discovery of the *plausibility* of the *reduction* of arithmetical content, as described in *Grundlagen*, and the logical *justification* of this content within a system of pure logic, as

[31] Since, for Frege, the reference of a term is determined by its sense, if definitions need not be reference-preserving, then they also need not be sense-preserving. Here, I shall focus on Benacerraf's argument for the claim Frege's definitions are not expected to preserve the reference of the numerals.

[32] Weiner uses the term "refer" as a technical term, such that a term refers when it is "appropriate for scientific use". On her account, Frege did think that the numerals express some content prior to his work, but that they did not have "scientific reference".

[33] [May and Wehmeier(2016)] make a similar point: "In general, Frege's criterion for the adequacy of definitions is holistic; it depends on what can be proven from the definition. Accordingly the adequacy of the definition of number is shown by the proof from it of the "basic laws of cardinal number"" [3fn.7].

shown in *Grundgesetze*.

## References


[Belnap(1993)] Nuel Belnap. On rigorous definitions. *Philosophical studies*, 72(2):115–146, 1993.

[Benacerraf(1965)] Paul Benacerraf. What numbers could not be. *The Philosophical Review*, 74(1): 47–73, 1965.

[Benacerraf(1981)] Paul Benacerraf. Frege: The last logicist. *Midwest Studies in Philosophy*, 6(1): 17–36, 1981.

[Blanchette(1994)] Patricia A Blanchette. Frege's reduction. *History and Philosophy of Logic*, 15(1): 85–103, 1994.

[Boddy(manuscript)] Rachel Boddy. Fruitful definitions, manuscript.

[Dummett(1991)] Michael AE Dummett. *Frege: Philosophy of mathematics*. Harvard University Press, 1991.

[Ebert and Rossberg (eds.)(2016)] Philip A Ebert and Marcus Rossberg (eds.). *Essays on Frege's Basic Laws of Arithmetic*. Oxford University Press, 2016.

[Floyd and Shieh(2001)] Juliet Floyd and Sanford Shieh. *Future pasts: the analytic tradition in twentieth-century philosophy*. Oxford University Press, 2001.

[Frege(1879)] Gottlob Frege. Begriffsschrift, eine der arithmetischen nachgebildete formelsprache des reinen denkens. halle. 1879. translated in van heijenoort j. begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, pages 3–82, 1879.

[Frege(1885)] Gottlob Frege. On formal theories of arithmetic. *Collected papers*, pages 203–250, 1885.

[Frege(1914)] Gottlob Frege. Logic in mathematics. *Posthumous writings*, pages 203–250, 1914.

[Frege and Austin(1980)] Gottlob Frege and John Langshaw Austin. *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number*. Northwestern University Press, 1980.

[Frege and MacGuinness(1984)] Gottlob Frege and Brian Francis MacGuinness. *Collected papers on mathematics, logic, and philosophy*. Blackwell, 1984.

[Frege et al.(2013)Frege, Ebert, and Rossberg] Gottlob Frege, Philip A Ebert, and Marcus Rossberg. *Gottlob Frege: Basic Laws of Arithmetic*, volume 1. Oxford University Press, 2013.

[Goldfarb(2001)] W Goldfarb. Frege's conception of logic. In: *Future pasts: The analytic tradition in twentieth century philosophy*, 2001.

[Heck(1993)] Richard G Heck. The development of arithmetic in frege's grundgesetze der arithmetik. *The Journal of Symbolic Logic*, 58(02):579–601, 1993.

[Horty(2007)] John Horty. *Frege on definitions: A case study of semantic content*. Oxford University Press, 2007.

[Jeshion(2001)] Robin Jeshion. Frege's notions of self-evidence. *Mind*, 110(440):937–976, 2001.

[May and Wehmeier(2016)] R May and K Wehmeier. The proof of hume's principle. *Ebert and Rossberg*, 2016.

[May(2012)] Robert May. What frege's theory of identity is not. *Thought: A Journal of Philosophy*, 1 (1):41–48, 2012.

[Shieh(2008)] Sanford Shieh. Frege on definitions. *Philosophy Compass*, 3(5):992–1012, 2008.

[Tappenden(1995)] Jamie Tappenden. Extending knowledge and fruitful concepts: Fregean themes in the foundations of mathematics. *Noûs*, 29(4):427–467, 1995.

[Weiner(1984)] Joan Weiner. The philosopher behind the last logicist. *The Philosophical Quarterly (1950-)*, 34(136):242–264, 1984.

[Weiner(1990)] Joan Weiner. *Frege in perspective*. Cornell University Press, 1990.